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A central limit theorem for marginally coupled designs

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A B S T R A C T

In this paper, we derive a central limit theorem for marginally coupled designs that are intended for computer experiments with both qualitative and quantitative factors. This result is useful for establishing confidence intervals for estimators in various statistical applications.

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1. Introduction

Computer experiments are widely used in many fields for studying complex physical phenomena that might otherwise be too time-consuming, costly, or even impossible to conduct. The standard framework for computer experiments always assumes that input factors are quantitative. However, in many computer experiments, some factors are qualitative by nature. For example, Schmidt et al. (2005) described a data center computer experiment that involved qualitative factors (such as diffuser location and hot-air-return-vent location) and quantitative factors (such as rack power and diffuser flow rate).

For constructing designs of computer experiments with qualitative and quantitative factors, Qian (2012) constructed a class of sliced space-filling designs, called sliced Latin hypercube designs (SLHDs). These designs consist of slices of space-filling designs with each slice corresponding to a level combination of the qualitative factors. Thus, the run sizes of these designs can be very large, even for a moderate number of qualitative factors. Deng et al. (2015) proposed a new type of designs with economical run sizes, called marginally coupled designs (MCDs). And, He et al. (2017) introduced some methods for constructing MCDs with improved space-filling property in designs for quantitative factors.

The sampling properties of the designs are helpful in numerical integration, stochastic optimization and uncertainty quantification. However, Stein (1987), Owen (1992), Loh (1996), Ai et al. (2016) and Kong et al. (2017) studied the sampling properties of only those designs that involved quantitative factors. Qian (2012) and He and Qian (2016) derived the sampling properties and the central limit theorem of SLHDs for computer experiments with qualitative and quantitative factors, respectively. The sampling properties of MCDs are thus an important but unresolved issue. The objective of this paper is to derive the asymptotic variance of an estimator for the expectation of function of output variables and the corresponding

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central limit theorem for MCDs. In addition, we use the D_{value} defined as (4.1) to compare SLHDs with MCDs to demonstrate that MCDs perform better than SLHDs under model (1.1).

In this paper, we consider a computer model that neglects the interactions among qualitative factors and interactions among quantitative and qualitative factors. To be specific, let $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_p)$ and $\boldsymbol{x} = (x_1, \ldots, x_q)$ denote the p qualitative factors and q quantitative factors, respectively. The response $f(\boldsymbol{w})$ at the input value $\boldsymbol{w} = (\boldsymbol{\gamma}, \boldsymbol{x})$ is

$$f(\boldsymbol{w}) = \mu + \tau_1(\gamma_1) + \cdots + \tau_p(\gamma_p) + g(\boldsymbol{x}), \tag{1.1}$$

where μ denotes the gross mean; $\tau_i(\gamma_i)$ denotes the effect of the ith qualitative factor at the level γ_i, $i = 1, \ldots, p$; $g(\boldsymbol{x})$ represents the response for the quantitative factor at input $\boldsymbol{x}$, under the assumption that $E(g(\boldsymbol{x})) = 0$; $\text{var}(g(\boldsymbol{x})) = \sigma^2$; and $\boldsymbol{x}$ follows a uniform distribution on $[0, 1]^q$. In addition, the effect of the qualitative factors satisfies

$$\sum_{\boldsymbol{\gamma} \in S_i} \tau_i(\boldsymbol{\gamma}) = 0 \text{ for } i = 1, \ldots, p,$$

where S_i denotes the set of the collection of all levels of the ith qualitative factor.

This paper is organized as follows. Section 2 introduces some useful definitions, notation and related work. Section 3 derives the asymptotic variance of the estimator and the corresponding central limit theorem for MCDs. The numerical illustrations of the derived theoretical results are given in Section 4. Some concluding remarks are provided in Section 5. All the proofs are provided in the online Supplementary Materials to save space.

2. Notation, definitions and related work

An orthogonal array A of strength t is an $n \times p$ matrix in which the jth column has s_j distinct levels $\{1, 2, \ldots, s_j\}$, and for every $n \times t$ sub-matrix of A, each possible level combination appears equally often as rows (see Hedayat et al., 1999). When all s_j's are not equal, the orthogonal array is asymmetric and denoted by $OA(n, s_1^{p_1} \ldots s_k^{p_k}, t)$, where the first p_1 columns have s_1 levels, the next p_2 columns have s_2 levels, and so forth. If all s_j's are equal to s, the orthogonal array is symmetric and denoted by $OA(n, s^p, t)$, where p is the number of factors. Orthogonal arrays are very popular in fractional factorial designs.

A Latin hypercube of n runs for q factors, denoted by $L(n, q)$, is represented by an $n \times q$ matrix $L = (l_{ik})$, in which each column is a uniform permutation on $\{1, \ldots, n\}$ (i.e., randomly taking a permutation on the set $\{1, \ldots, n\}$, with all $n!$ possible permutations having equally probability), and all columns are independently obtained (see McKay et al., 1979). A Latin hypercube $L(n, q)$ can be generated from an $OA(n, s^q, t)$, by replacing the $r = n/s$ positions for level i with a random permutation of $\{(i-1)r + 1, \ldots, ir\}$, for $i = 1, \ldots, s$, and the resulting Latin hypercube is known as an OA-based Latin hypercube (see Tang, 1993).

An ordinary Latin hypercube design $X = (x_{ik})$ of n runs for q factors, denoted by $LHD(n, q)$, is generated by a Latin hypercube $L = (l_{ik})$ through

$$x_{ik} = \frac{l_{ik} - u_{ik}}{n}, \text{ for } i = 1, \ldots, n, \ k = 1, \ldots, q, \tag{2.1}$$

where the u_{ik}'s are independent $U(0, 1]$ random variables, and the u_{ik}'s are mutually independent of the l_{ik}'s. This class of design is popular because when X is projected onto any one dimension, precisely one point falls within one of the n equally spaced intervals of [0,1], as given by $[0, 1/n), [1/n, 2/n), \ldots, [(n-1)/n, 1]$. Different variants of Latin hypercube designs have been developed in the literature. Orthogonality and maximin distance are two commonly used design criteria (see Sun et al., 2009; Zhou and Xu, 2014; Sun and Tang, 2017; and the references therein for details).

A Latin hypercube L of $n = m\lambda$ runs is called a sliced Latin hypercube of λ slices if L can be expressed as $L = (L_1', \ldots, L_\lambda')'$, each L_i has m runs, and the m points in each column of L_i have exactly one point from each of the m equally sized sets $\{(j-1)\lambda + 1, \ldots, j\lambda\}$, $1 \le j \le m$. A sliced Latin hypercube design (SLHD) $X = (x_{ij})$ of n runs for q factors can be generated by a sliced Latin hypercube via (2.1). The whole design and each slice of the design can achieve maximum uniformity in any one-dimensional projection.

Let D_1 be an $OA(n, s^p, 2)$ and D_2 be a Latin hypercube design. Then, the design $D = (D_1, D_2)$ is called a marginally coupled design if the rows in D_2 corresponding to each level of any factor in D_1 form a small Latin hypercube design (see Deng et al., 2015). The D_1 and D_2 are designs for qualitative and quantitative factors, respectively. We use MCD(D_1, D_2) to denote this design hereafter. Example 1 below provides an MCD(D_1, D_2) of 8 runs for two qualitative factors and two quantitative factors.

Example 1. Table 1 presents an example of MCD(D_1, D_2) for two 2-level qualitative factors, $\boldsymbol{\gamma} = (\gamma_1, \gamma_2)$, and two quantitative factors, $\boldsymbol{x} = (x_1, x_2)$. Fig. 1 displays scatter plots of x_1 versus x_2. The rows of D_2, which correspond to levels of γ_1 and γ_2, are listed on the left and right sides, respectively. Projected onto x_1 or x_2, only one '1' or '2' is located in each of the four intervals of $[0, 1/4), [1/4, 2/4), [2/4, 3/4), [3/4, 1]$. Thus, for each level of any factor in D_1, the corresponding rows of D_2 possess maximum uniformity in any one-dimensional projection.

For subsequent development, we review an algorithm for constructing the MCD(D_1, D_2) with n runs, p qualitative factors, and q quantitative factors through an asymmetric $OA(n, s^p(n/s), 2)$ and an $OA(n/s, s_1^q, 2)$, where s_1 and s can differ. Suppose that there exist an $OA(n, s^p(n/s), 2)$ and an $OA(n/s, s_1^q, 2)$, denoted by A and B. The algorithm is as follows (see He et al., 2017).

Table 1
A marginally coupled design MCD(D_1, D_2).

D_1		D_2	
γ_1	γ_2	x_1	x_2
1	1	0.0904	0.3202
1	1	0.3692	0.7023
1	2	0.8125	0.1543
1	2	0.5221	0.9006
2	2	0.1631	0.4766
2	2	0.4604	0.5638
2	1	0.8812	0.0693
2	1	0.6875	0.7942

Where D_1 is an $OA(8, 2^2, 2)$ and D_2 is an $LHD(8, 2)$.

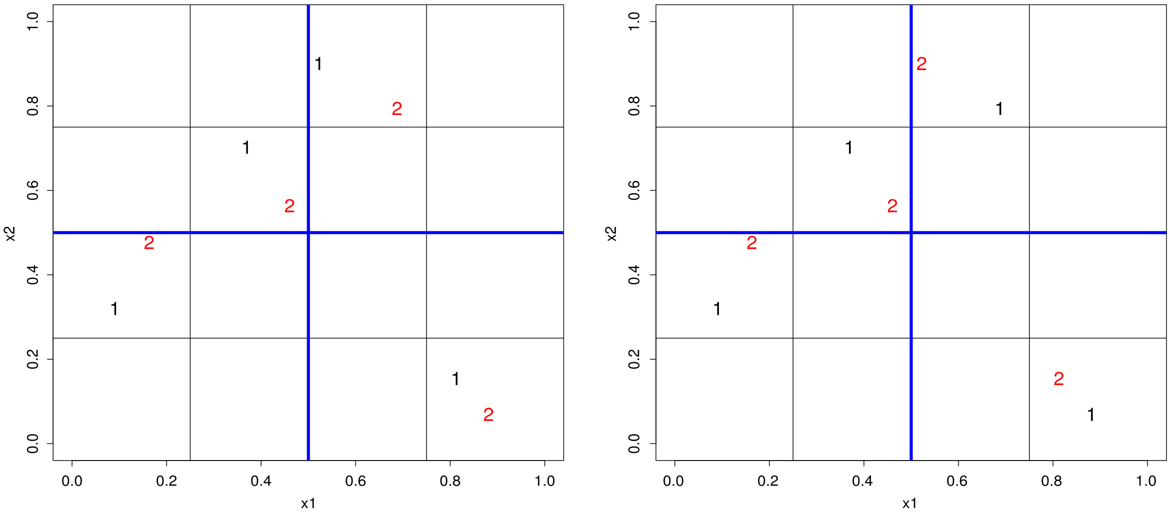

Fig. 1. Scatter plots of x_1 versus x_2 in Example 1, where the rows of D_2 correspond to levels 1 and 2 of γ_1 (left) and γ_2 (right), respectively.

Algorithm 1.

Step 1. Derive an OA-based $L(n/s, q)$, say L, based on the orthogonal array B.

Step 2. Obtain an $n \times q$ matrix $\tilde{D}_2$ by replacing the levels $1, \ldots, n/s$ of the last column of A with the 1st, ..., and the (n/s)-th row of the L obtained in Step 1.

Step 3. Construct an $n \times q$ matrix $\hat{D}_2$ based on $\tilde{D}_2$ from Step 2 by replacing the s entries with level i in each column of $\tilde{D}_2$ by a random permutation of $\{(i-1)s+1, \ldots, is\}$ for $i = 1, \ldots, n/s$.

Step 4. Generate an $LHD(n, q)$, D_2, based on $\hat{D}_2$ by (2.1).

Let D_1 be the first p columns of A. Then, $D = (D_1, D_2)$ is a marginally coupled design, where D_2 is generated by Algorithm 1.

3. A central limit theorem for marginally coupled designs

In this section, we first derive the asymptotic variance of an estimator for the expectation of function of output variables for a general MCD(D_1, D_2), with D_1 being an $OA(n, s^p, 2)$ and D_2 being an $LHD(n, q)$. Then, a central limit theorem for MCDs is derived to show that the estimator has a limiting normal distribution.

Suppose that the model is defined as (1.1). Let d_{lc} be the collection of the rows of the MCD(D_1, D_2) that correspond to the cth level of the lth qualitative factor for $l = 1, \ldots, p,\ c = 1, \ldots, s$. According to the definition of MCD(D_1, D_2), d_{lc} is a small Latin hypercube design with m runs, where $m = n/s$. The notation $i \in d_{lc}$ indicates that the ith run $\boldsymbol{w}_i = (\boldsymbol{\gamma}_i, \boldsymbol{x}_i)$ belongs to d_{lc}. We are interested in the linear combination of $\mu_{l1}, \ldots, \mu_{ls}$,

$$\eta = \sum_{c=1}^{s} \lambda_c \mu_{lc}, \text{ where } \mu_{lc} = \frac{1}{m} \sum_{i \in d_{lc}} E(f(\boldsymbol{w}_i)). \tag{3.1}$$

η is the gross mean of the response when $\lambda_c = 1/s$ and represents a treatment contrast of the lth qualitative factor when $\sum_{c=1}^{s} \lambda_c = 0$ (see Mukerjee and Wu, 2006).

Lemma 1. *Based on the assumptions of model (1.1) and the design of MCD(D_1, D_2), we have*

(1) $\mu_{lc} = \mu + \tau_l(c), l = 1, \ldots, p,\ c = 1, \ldots, s;$
(2) $\hat{\mu}_{lc} = \frac{1}{m} \sum_{i \in d_{lc}} f(\boldsymbol{w}_i)$ *is an unbiased estimator of* μ_{lc}*;*
(3) $\hat{\eta} = \sum_{c=1}^{s} \lambda_c \hat{\mu}_{lc}$ *is an unbiased estimator of* η*.*

To derive the variances of $\hat{\mu}_{lc}$ and $\hat{\eta}$ for MCD(D_1, D_2), we first introduce the functional analysis of variance (ANOVA) decomposition (see Owen, 1992; Loh, 1996). Let F be the uniform measure on $[0, 1]^q$ with $dF = \prod_{k=1}^{q} dx_k$ and $dF_{-k} = \prod_{l \neq k} dx_l$. Suppose that $g : [0, 1]^q \to \mathscr{R}$ is a continuous function on $[0, 1]^q$ with mean $\int g(\boldsymbol{x})dF = 0$ and variance $\int g^2(\boldsymbol{x})dF = \sigma^2$. Then, $g(\boldsymbol{x})$ can be decomposed as

$$g(\boldsymbol{x}) = \sum_{k=1}^{q} g_{-k}(x_k) + r(\boldsymbol{x}), \tag{3.2}$$

where $g_{-k}(x_k) = \int g(\boldsymbol{x})dF_{-k}$, is the main effect function for the kth input variable, and $r(\boldsymbol{x})$ is the residual.

Note that $\int g_{-k}(x_k)dF_k = 0$, $\int r(\boldsymbol{x})dF_{-k} = 0$. It can be verified that $\int g_{-k}(x_k)g_{-l}(x_l)dF = 0$ for $k \neq l$ and $\int g_{-k}(x_k)r(\boldsymbol{x})dF = 0$. Thus, the variance of $g(\boldsymbol{x})$ can be decomposed as

$$\sigma^2 = \sum_{k=1}^{q} \int g_{-k}^2(x_k)dx_k + \int r^2(\boldsymbol{x})d\boldsymbol{x}. \tag{3.3}$$

Next, we provide the covariance between $f(\boldsymbol{w}_i)$ and $f(\boldsymbol{w}_j)$ for $i, j \in d_{lc}$ and $i \neq j$.

Theorem 1. *Suppose that* $f(\boldsymbol{w}_i)$ *is the model defined as (1.1), which is evaluated on an MCD(D_1, D_2), and* $g : [0, 1]^q \to \mathscr{R}$ *is a continuous function. Then, as* $n = ms \to +\infty$ *with* s *fixed, for* $i, j \in d_{lc}$ *and* $i \neq j$*, we have*

$$\mathrm{cov}\left(f(\boldsymbol{w}_i), f(\boldsymbol{w}_j)\right) = -m^{-1} \sum_{k=1}^{q} \int_0^1 g_{-k}^2(x_k)dx_k + o(m^{-1}). \tag{3.4}$$

According to Theorem 1, we can obtain the variances of $\hat{\eta}$ and $\hat{\mu}_{lc}$ for $l = 1, \ldots, p,\ c = 1, \ldots, s$.

Theorem 2. *Suppose that* $f(\boldsymbol{w}_i)$ *is the model defined as (1.1), which is evaluated on an MCD(D_1, D_2), and* $g : [0, 1]^q \to \mathscr{R}$ *is a continuous function. Then, as* $n = ms \to +\infty$ *with* s *fixed, we have*

(1) $\mathrm{var}\,(\hat{\mu}_{lc}) = m^{-1}\sigma^2 - m^{-1} \sum_{k=1}^{q} \int_0^1 g_{-k}^2(x_k)dx_k + o(m^{-1});$
(2) $\mathrm{var}\,(\hat{\eta}) = m^{-1} \sum_{c=1}^{s} \lambda_c^2 \sigma^2 - m^{-1} \sum_{c=1}^{s} \sum_{k=1}^{q} \lambda_c^2 \int_0^1 g_{-k}^2(x_k)dx_k + o(n^{-1}).$

By Theorem 2 and (3.3), the variances of $\hat{\mu}_{lc}$ and $\hat{\eta}$ can be simplified as follows:

$$\mathrm{var}\,(\hat{\mu}_{lc}) = m^{-1} \int r^2(\boldsymbol{x})d\boldsymbol{x} + o(m^{-1}) \text{ and } \mathrm{var}\,(\hat{\eta}) = m^{-1} \sum_{c=1}^{s} \lambda_c^2 \int r^2(\boldsymbol{x})d\boldsymbol{x} + o(n^{-1}). \tag{3.5}$$

With these preparations, we now turn to discuss the limiting distributions of $\hat{\mu}_{lc}$ and $\hat{\eta}$.

Theorem 3. *Suppose that* $f(\boldsymbol{w}_i)$ *is the model defined as (1.1), which is evaluated on an MCD(D_1, D_2), and* $g : [0, 1]^q \to \mathscr{R}$ *is a continuous function. Then, as* $n = ms \to +\infty$ *with* s *fixed, for* $l = 1, \ldots, p,\ c = 1, \ldots, s$*, we have*

$$\sqrt{m}(\hat{\mu}_{lc} - \mu_{lc}) \to N\left(0,\ \int_0^1 r^2(\boldsymbol{x})d\boldsymbol{x}\right),\ \sqrt{m}(\hat{\eta} - \sum_{c=1}^{s} \lambda_c \mu_{lc}) \to N\left(0,\ \sum_{c=1}^{s} \lambda_c^2 \int_0^1 r^2(\boldsymbol{x})d\boldsymbol{x}\right).$$

4. Numerical illustrations

In this section, we provide two numerical examples to corroborate the theoretical results derived in the previous section. Obviously, the variance of the random error is unimportant because different experiment designs have the same random error variance. In addition, we use a standard variance of the design (see Kiefer, 1961)

$$D_{\text{value}} = n \times \mathrm{var}\,(\hat{\eta}), \tag{4.1}$$

to compare designs with different run sizes n, and the smaller D_{value} is, the better the design is.

Table 2
The comparison of SLHD and MCD(D_1, D_2) for Example 2.

	MCD(D_1, D_2)	SLHD
n	32	64
var ($\hat{\eta}$)	0.0303	0.0259
var ($\hat{\mu}_{11}$)	0.0330	0.0453
var ($\hat{\mu}_{12}$)	0.0325	0.0459
D_{value}	0.9696	1.6576

Table 3
Comparison of variances between MCD(D_1, D_2) and SLHD for Example 3.

	MCD(D_1, D_2)	SLHD
n	27	81
var ($\hat{\eta}$)	0.0089	0.0050
var ($\hat{\mu}_{11}$)	0.0363	0.0529
var ($\hat{\mu}_{12}$)	0.0361	0.0523
var ($\hat{\mu}_{13}$)	0.0363	0.0509
D_{value}	0.2403	0.4050

Example 2. Consider a computer experiment with four 2-level qualitative factors and four quantitative factors, which is simulated as follows:

$$f(\boldsymbol{w}) = 10 + \tau_1(\gamma_1) + \tau_2(\gamma_2) + \tau_3(\gamma_3) + \tau_4(\gamma_4) + g(\boldsymbol{x}),$$

where $\gamma_i \in \{1, 2\}, i = 1, 2, 3, 4, \tau_1 = (-1, 1)', \tau_2 = (8, -8)', \tau_3 = (10, -10)', \tau_4 = (-15, 15)'$, and $g(\boldsymbol{x}) = 2/3e^{x_1+x_2} - x_4\sin(x_3) + x_3 - 2.23$. The distribution of $\boldsymbol{x}$ is the uniform measure on $[0, 1]^4$. $g(\boldsymbol{x})$ is also used as a simulation function in Cox et al. (2001) for computer experiments that contain only quantitative factors.

The experiment is conducted by MCD(D_1, D_2) 1000 times, where D_1 is an $OA(32, 2^4, 4)$ and D_2 is an $LHD(32, 4)$ (The MCD(D_1, D_2) can be generated by Algorithm 1 based on an asymmetric $OA(32, 2^4 16^1, 2)$ and an $OA(16, 2^4, 2)$). Because there are 16 level combinations of qualitative factors, if we use an SLHD, the design should contain 16 slices with each slice containing m runs. Consider $m = 2, 3,$ and 6, then the run sizes of SLHDs are 32, 48, and 96. The experiment is also conducted via SLHD 1000 times. The values of $\hat{\eta}$ (here, we use $l = 1$ and $\lambda_1 = \lambda_2 = 1/2$ without loss of generality) and $\hat{\mu}_{1c}$ can be obtained based on experiments.

Table 2 compares the variances of $\hat{\eta}, \hat{\mu}_{11}, \hat{\mu}_{12}, D_{\text{value}}$ for MCD(D_1, D_2) and SLHD. Note that D_{value} is the standard variance defined in (4.1) for a design with n runs. We can draw the following conclusions: (i) For the same run size, as shown in columns 2 and 3, MCD(D_1, D_2) performs better than SLHD under var ($\hat{\eta}$), var ($\hat{\mu}_{11}$), var ($\hat{\mu}_{12}$) and D_{value}. (ii) When the run size of SLHD increases to 48, which corresponds to column 4, the design can achieve the same variance reduction as a 32-run MCD(D_1, D_2) for $\hat{\eta}$, but MCD(D_1, D_2) still performs better under var ($\hat{\mu}_{1i}$) and D_{value}. (iii) Furthermore, when the run size of SLHD increases to 96, as shown in column 5, even though it has the same or slightly smaller var ($\hat{\eta}$), var ($\hat{\mu}_{11}$), and var ($\hat{\mu}_{12}$), the run size is 3 times as large as that of a 32-run MCD(D_1, D_2). As a result, it can be concluded that MCD(D_1, D_2) performs better than SLHD for the same run size and is a good substitute with an economical run size for SLHD.

For MCD(D_1, D_2), based on the results of the experiment in Example 2, the density plot and asymptotic theoretical distribution of $\sqrt{m}(\hat{\eta} - \sum_{c=1}^{s} \lambda_c \mu_{lc})$ are given in Fig. 2. It is clear that the density plot (dotted curve) is close to the corresponding asymptotic theoretical distribution (solid curve). This figure corroborates the theoretical result of Theorem 3.

Example 3. Consider a computer experiment with three 3-level qualitative factors and four quantitative factors, which is simulated as follows:

$$g(\boldsymbol{w}) = 10 + \tau_1(\gamma_1) + \tau_2(\gamma_2) + \tau_3(\gamma_3) + g(\boldsymbol{x}),$$

where $\gamma_i \in \{1, 2, 3\}, i = 1, 2, 3, \tau_1 = (0, -2/3, 2/3)', \tau_2 = (0, -1, 1)', \tau_3 = (8, -4, -4)'$, and $g(\boldsymbol{x}) = x_1(\sqrt{1 + (x_2 + x_3^2)x_4/x_1} - 1)/2 + x_1 + 3x_4 - 2.16$. The distribution of $\boldsymbol{x}$ is the uniform measure on $[0, 1]^4$. $g(\boldsymbol{x})$ is also used as a simulation function in Cox et al. (2001) for computer experiments that contain only quantitative factors.

For this experiment, we use MCD(D_1, D_2) and three SLHDs to estimate $\eta, \mu_{11}, \mu_{12}, \mu_{13}$ over 1000 replicates for each design. MCD(D_1, D_2) has 27 runs and is constructed via Algorithm 1 based on an $OA(27, 3^3 9^1, 2)$ and an $OA(9, 3^4, 2)$. The three SLHDs contain 27 slices, which have 27, 54, and 108 runs with 1, 2, and 4 runs in each slice. Table 3 compares the variances of $\hat{\eta}, \hat{\mu}_{11}, \hat{\mu}_{12}, \hat{\mu}_{13}$ for MCD(D_1, D_2) and the three SLHDs, with $l = 1$ and $\lambda_1 = \lambda_2 = \lambda_3 = 1/3$. Again, MCD($D_1, D_2$) with an economical run size achieves the best performance, and conclusions similar to those based on Table 2 are obtained. Fig. 3 presents the density plot and asymptotic theoretical distribution of $\sqrt{m}(\hat{\eta} - \sum_{c=1}^{s} \lambda_c \mu_{lc})$ for MCD(D_1, D_2). This figure corroborates the theoretical result of Theorem 3.

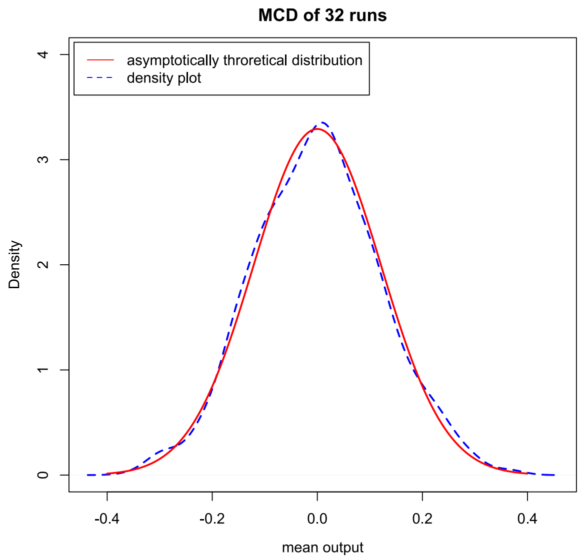

Fig. 2. Density plot (dotted curve) of $\sqrt{m}(\hat{\eta} - \sum_{c=1}^{s} \lambda_c \mu_{1c})$ in Example 2 based on MCD and the corresponding asymptotic theoretical distribution (solid curve).

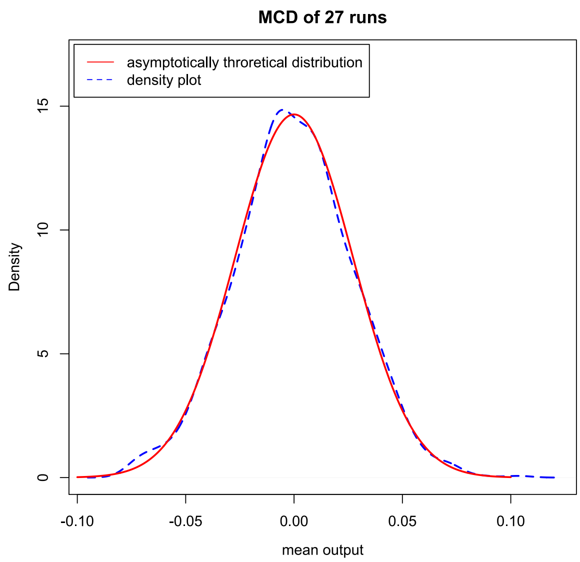

Fig. 3. Density plot (dotted curve) of $\sqrt{m}(\hat{\eta} - \sum_{c=1}^{s} \lambda_c \mu_{1c})$ in Example 3 based on MCD and the corresponding asymptotic theoretical distribution (solid curve).

5. Concluding remarks

We provided a theoretical foundation for the use of MCDs to conduct computer experiments with both qualitative and quantitative factors under model (1.1). Two examples comparing SLHDs and MCDs are presented. According to the comparisons, MCDs are a good substitute for SLHDs when the cost of experimentation is too expensive. Model (1.1) does not consider the interactions among qualitative factors or interactions among quantitative and qualitative factors, and the results obtained under a general model will be studied in the future.

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Appendix A. Supplementary data

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.spl.2018.11.018.

References

Ai, M.Y., Kong, X.S., Li, K., 2016. A general theory for orthogonal array based Latin hypercube sampling. Statist. Sinica 26, 761–777.
Cox, D.D., Park, J.S., Singer, C.E., 2001. A statistical method for tuning a computer code to a data base. Comput. Statist. Data Anal. 37, 77–92.
Deng, X.W., Hung, Y., Lin, C.D., 2015. Design for computer experiments with qualitative and quantitative factors. Statist. Sinica 25, 1567–1581.
He, Y.Z., Lin, C.D., Sun, F.S., 2017. On construction of marginally coupled designs. Statist. Sinica 27, 665–683.
He, X., Qian, P.Z.G., 2016. A central limit theorem for nested or sliced Latin hypercube designs. Statist. Sinica 26, 1117–1128.
Hedayat, A.S., Sloane, N.J.A., Stufken, J., 1999. Orthogonal Array: Theory and Applications. Springer, New York.
Kiefer, J., 1961. Optimum designs in regression problems, II. Ann. Math. Stat. 32, 298–325.
Kong, X.S., Ai, M.Y., Mukerjee, R., 2017. Central limit theorems for four new types of U-designs. Statistics 51, 655–667.
Loh, W.L., 1996. On Latin hypercube sampling. Ann. Statist. 24, 2058–2080.
McKay, M.D., Beckman, R.J., Conover, W.J., 1979. A comparison of three methods for selecting values of input variables in the analysis of output from a computer code. Technometrics 21, 239–245.
Mukerjee, R., Wu, C.F.J., 2006. A Modern Theory of Factorial Designs. Springer, New York.
Owen, A.B., 1992. A central limit theorem for Latin hypercube sampling. J. R. Stat. Soc. Ser. B Stat. Methodol. 54, 541–551.
Qian, P.Z.G., 2012. Sliced Latin hypercube designs. J. Amer. Statist. Assoc. 107, 393–399.
Schmidt, R.R., Cruz, E.E., Iyengar, M.K., 2005. Challenges of data center thermal management. IBM J. Res. Dev. 49, 709–723.
Stein, M., 1987. Large sample properties of simulations using Latin hypercube sampling. Technometrics 29, 143–151.
Sun, F.S., Liu, M.Q., Lin, D.K.J., 2009. Construction of orthogonal Latin hypercube designs. Biometrika 96, 971–974.
Sun, F.S., Tang, B.X., 2017. A method of constructing space-filling orthogonal designs. J. Amer. Statist. Assoc. 112, 683–689.
Tang, B.X., 1993. Orthogonal array-based Latin hypercubes. J. Amer. Statist. Assoc. 88, 1392–1397.
Zhou, Y.D., Xu, H.Q., 2014. Space-filling fractional factorial designs. J. Amer. Statist. Assoc. 109, 1134–1144.